

Covariant Symplectic Structure and Conserved Charges of New Massive Gravity

Gökhan Alkaç* and Deniz Olgu Devecioğlu†

*Department of Physics, Faculty of Arts and Sciences,
Middle East Technical University, 06800, Ankara, Turkey*

(Dated: April 13, 2012)

Abstract

We show that the symplectic current obtained from the boundary term, which arises in the first variation of a local diffeomorphism invariant action, is covariantly conserved for any gravity theory described by that action. Therefore, a Poincaré invariant two-form can be constructed on the phase space, which is shown to be closed without reference to a specific theory. Finally, we show that one can obtain a charge expression for gravity theories in various dimensions, which plays the role of the Abbott-Deser-Tekin charge for spacetimes with nonconstant curvature backgrounds, by using the diffeomorphism invariance of the symplectic two-form. As an example, we calculate the conserved charges of some solutions of new massive gravity and compare the results with previous works.

PACS numbers: 04.20.Cv, 04.20.Fy, 04.50.Kd

*Electronic address: alkac@metu.edu.tr

†Electronic address: dedeveci@metu.edu.tr

I. INTRODUCTION

The inherent nature of Hamiltonian formulation seems to shelter a conflict with the sacred property of general covariance by the choice of time coordinate. This very fact is a major obstacle in the definition of conserved charges especially in gravity theories. The Arnowitt-Deser-Misner (ADM) mass [1] is a good example of this hindrance. One has to deal with the decomposition of spacetime into spacelike hypersurfaces parametrized by the time coordinate. Achieving this for higher curvature gravity theories is obviously a tedious task to perform.

The aim of this paper is to circumvent this difficulty by employing the construction of [2–4] which simply builds up the phase space from the solutions of the classical equations. The symplectic two-form identified through this way contains all of the relevant properties of the phase space without the need for defining momenta. Having constructed the symplectic structure, the diffeomorphism invariance of the symplectic two-form lets one find a closed expression to compute the conserved charges of the solutions of the theory, which is of paramount importance to understand the thermodynamical properties of the solutions. The most important result we will prove in this paper is the equivalence of this charge expression to the Abbott-Deser-Tekin (ADT) [5–7] charge when the diffeomorphisms are restricted to be the isometries of the background spacetime.

For topologically massive gravity (TMG) [8] the symplectic two-form and the conserved charges were given in [9]. In this work we focus on a three dimensional gravitational theory that has attracted considerable attention recently. This theory, termed as new massive gravity (NMG) [10, 11], is obtained by adding a particular higher curvature term ($\alpha R^2 + \beta R_{ab}^2$ with the constraint $8\alpha + 3\beta = 0$) to the Einstein-Hilbert action, which makes the theory tree-level unitary [12] but not renormalizable [13]. It is a valuable toy model for our purposes since many interesting solutions with AdS_3 and arbitrary backgrounds have appeared in the literature [14–18].

The outline of the paper is as follows: Sec. II starts with the definition of the fundamental objects on the phase space and continues with the construction of the symplectic two-form ω , for the theories derived from the action

$$I = \int d^D x \sqrt{|g|} \left(\frac{1}{\kappa} (R + 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2 \right).$$

We end up the section with the discussion of the gauge invariance of ω . In Sec. III, we find an

expression for the conserved charges of these theories and show its equivalence to the ADT charge. Section IV is devoted to the computation of the energy and angular momentum of some solutions of NMG using the formulas derived in Sec. III.

Our conventions are as follows: The signature of the metric is $(-, +, \dots, +)$. The Riemann tensor is defined through $[\nabla_a, \nabla_b]V_c = R_{abc}{}^d V_d$ and $R_{ab} := R^c{}_{acb}$. For the symmetrization and antisymmetrization of tensors, the factors and signs are chosen so that e.g. $T_{(ab)} \equiv \frac{1}{2}(T_{ab} + T_{ba})$, $T_{[ab]} \equiv \frac{1}{2}(T_{ab} - T_{ba})$.

II. THE CONSTRUCTION OF THE SYMPLECTIC TWO-FORM

First we start by summarizing the covariant canonical formulation of classical theories developed by [2–4], in a way that manifestly preserves relevant symmetries of the theory. Before delving into details, let us recall the usual canonical formalism of a theory. One starts with a $2N$ dimensional smooth manifold Z endowed with a two-form given as

$$\omega = dp_i \wedge dq^i, \tag{1}$$

where q^i and p_i are introduced as generalized coordinates and generalized momenta, respectively, and $i = 1, \dots, N$. Furthermore, ω is closed ($d\omega = 0$) and nondegenerate, i.e. when ω is written as a $2N \times 2N$ matrix, it has an inverse. This closed two-form ω on Z is called the symplectic two-form.

In order to develop and use this structure in geometrical theories derived from an action, we need to follow a somewhat different route from the usual approach discussed above, since choosing p_i and q^i as coordinates of the phase space Z would destroy the general covariance (by the choice of time coordinate). One should construct the phase space Z from the solutions of the equations of motion to achieve a manifestly covariant structure. Since classical solutions of any physical theory are in one-to-one correspondence with the initial values of p_i and q^i , we define our phase space as the space of solutions of the classical equations as suggested by [3]. By this way, starting from an arbitrary Lagrangian, the phase space Z will follow from the manifold of field configurations. Our next step is to define the fundamental objects on Z for geometrical construction of the phase space.

A. Fundamental objects on the phase space

We assume that the gravitational field equations are derived from the variation of a generic local gravity action that is a functional in metric g , Riemann tensor R and/or its contraction and covariant derivatives¹

$$S = \int d^D x \sqrt{|g|} \mathcal{L}(g, R, \nabla R, R^2, \dots). \quad (2)$$

Under first order variation, (2) can be written as

$$\delta S = \int d^D x \sqrt{|g|} \Phi \delta g + \int d^D x \partial \Lambda(g, \delta g, \nabla \delta g \dots), \quad (3)$$

where $\Phi = 0$ describes the field equations and $\partial \Lambda$ is a partial derivative of some boundary term with respect to the spacetime coordinates.

Let g be a solution of the field equations i.e. $\Phi(g) = 0$. The functions on Z , denoted by $g(x)$, take a spacetime point x and map it into a $D \times D$ real matrix $g(x)$. For the vectors, consider an arbitrary and small variation in the metric $\tilde{g} = g + \delta g$. When this is inserted into the field equations, it yields $\tilde{\Phi} = \Phi + \delta \Phi$. Here $\delta \Phi$ are obviously the linearized field equations. The vectors on Z can be defined as the variations δg that solve $\delta \Phi = 0$. With vectors in hand, the corresponding differential one-forms are easy to construct. A one-form, $\delta g(x)$, is the mapping from the vector δg to a $D \times D$ real matrix $\delta g(x)$, which is the vector evaluated at a spacetime point x . We can generalize this notion to construct more general p -forms as the “wedge functions” of the one-forms $\delta g(x)$

$$\Omega = \int dx_1 \cdots dx_p \Theta(x_1, \dots, x_p) \delta g(x_1) \wedge \cdots \wedge \delta g(x_p), \quad (4)$$

where $\Theta(x_1, \dots, x_p)$ is a zero-form on Z and \wedge is an anticommuting product. We can define an exterior derivative operator δ that maps p -forms to $(p+1)$ -forms as follows

$$\delta \Omega = \int dx_0 dx_1 \cdots dx_p \frac{\delta \Theta(x_1, \dots, x_p)}{\delta g(x_0)} \delta g(x_0) \wedge \delta g(x_1) \wedge \cdots \wedge \delta g(x_p), \quad (5)$$

where $\frac{\delta \Theta(x_1, \dots, x_p)}{\delta g(x_0)}$ is the functional derivative of Θ with respect to $g(x)$. One can easily check that this operator obeys the modified Leibniz rule and the celebrated Poincaré lemma. This construction and notation was due to [3], although one could also analyze the same problem in a different approach [4, 19].

¹ For the sake of simplicity, we drop the indices on all tensorial quantities discussed in this section.

B. The symplectic current and the symplectic two-form

We are now ready to construct a symplectic two-form for the theories described by (2). First let us reconsider (3) within the context of the formalism we have reviewed in the previous section. The variation of the action δS can be viewed as a one-form on Z (note that $\Lambda^a(x)$ contains δg_{ab} and all of the other relevant quantities such as $\delta \Gamma_{bc}^a$, δR_{ab} , etc.). The key identity, upon which the definition of covariantly conserved symplectic current is based, can be obtained from the exterior derivative of (3), which will vanish by the Poincaré Lemma

$$\delta^2 S = \int d^D x \sqrt{|g|} \delta \Phi_{ab} \wedge \delta g^{ab} - \frac{1}{2} \int d^D x \sqrt{|g|} \Phi_{ab} \delta g^{ab} \wedge \delta \ln |g| + \int d^D x \partial_a \delta \Lambda^a = 0, \quad (6)$$

where $\delta \ln |g| = g^{ab} \delta g_{ab} = -g_{ab} \delta g^{ab}$. The first two integrals vanish on-shell and the third one implies that

$$-\delta^2 S = \int d^D x \sqrt{|g|} \nabla_a J^a = 0, \quad (7)$$

where $J^a \equiv -\delta \Lambda^a / \sqrt{|g|}$ is the “symplectic current”. We emphasize that this result holds on-shell for any theory derived from (2) and clarifies the definitions of J^a given in [3, 9].

Using (7), one can construct the following Poincaré invariant two-form since the covariant divergence of the symplectic current vanishes ($\nabla_a J^a = 0$)

$$\omega = \int_{\Sigma} d\Sigma_a \sqrt{|g|} J^a, \quad (8)$$

where Σ is a $(D-1)$ -dimensional spacelike hypersurface. Darboux’s theorem guarantees that this is the sought-after symplectic two-form of the theory if ω is also closed, which can be shown by taking the exterior derivative of (8)

$$\delta \omega = \int_{\Sigma} d\Sigma_a (\delta \sqrt{|g|} \wedge J^a + \sqrt{|g|} \delta J^a). \quad (9)$$

To evaluate the second term in (9), we now appeal to the exterior derivative of (6)

$$\begin{aligned} \delta^3 S &= \int d^D x \sqrt{|g|} \left(\frac{1}{2} \delta \ln |g| \wedge \delta \Phi_{ab} \wedge \delta g^{ab} - \frac{1}{2} \delta \Phi_{ab} \wedge \delta g^{ab} \wedge \delta \ln |g| \right) \\ &\quad + \int d^D x \sqrt{|g|} \left(\frac{1}{2} \delta \ln |g| \wedge \nabla_a J^a + \delta(\nabla_a J^a) \right) = 0, \end{aligned} \quad (10)$$

the first two terms cancel each other. Thus we obtain

$$\delta^3 S = \frac{1}{2} \int d^D x \sqrt{|g|} \delta \ln |g| \wedge \nabla_a J^a + \int d^D x \sqrt{|g|} (\nabla_a \delta J^a + \delta \Gamma^b_{ab} \wedge J^a) = 0, \quad (11)$$

from which an important relation follows

$$\int_{\Sigma} d\Sigma_a \sqrt{|g|} \delta J^a = -\frac{1}{2} \int_{\Sigma} d\Sigma_a \sqrt{|g|} J^a \wedge \delta \ln|g|. \quad (12)$$

By virtue of (12) and bearing in mind that J^a is an anticommuting two-form, we see that (9) vanishes. It should be noted that this result holds without the use of the field equations, unlike the vanishing covariant divergence of J^a that is valid only on-shell. This result was obtained for general relativity and TMG by means of detailed calculations [3, 9]. Here we have given a completely general proof applicable to the current J^a derived from any local action (2).

C. The gauge invariance

Finally, one must show that the symplectic two-form is also gauge invariant in the space of classical solutions Z and in the quotient space $\bar{Z} = Z/G$, where G denotes the group of diffeomorphisms ($x^a \rightarrow x^a + \xi^a$). The former is trivial since all constituents of ω transform like tensors. For the latter, we need to find out how ω transforms under the following transformation

$$\delta g_{ab} \rightarrow \delta g_{ab} + \nabla_a \xi_b + \nabla_b \xi_a, \quad (13)$$

where ξ is asymptotic to a Killing vector field at the boundary of the spacetime. Being a function of δg_{ab} , the transformation of one-forms will follow from (13) easily. Some of the basic quantities transform as

$$\delta \Gamma^a_{bc} \rightarrow \delta \Gamma^a_{bc} + R^a_{ec} \xi^e_b + \nabla_c \nabla_b \xi^a, \quad (14)$$

$$\delta R_{ab} \rightarrow \delta R_{ab} + \xi^c \nabla_c R_{ab} + R_{ad} \nabla_b \xi^d + R_{bd} \nabla_a \xi^d. \quad (15)$$

As a general rule for a tensor $T_{a\dots}{}^{b\dots}$, which is a function of δg_{ab} or its covariant derivatives, the transformation reduces to

$$\begin{aligned} \delta T_{a\dots}{}^{b\dots} &\rightarrow \delta T_{a\dots}{}^{b\dots} + \xi^c \nabla_c T_{a\dots}{}^{b\dots} + T_{d\dots}{}^{b\dots} \nabla_a \xi^d + \dots - T_{a\dots}{}^{d\dots} \nabla^b \xi_d + \dots \\ &= \delta T_{a\dots}{}^{b\dots} + \mathcal{L}_{\xi} T_{a\dots}{}^{b\dots}, \end{aligned} \quad (16)$$

where \mathcal{L}_{ξ} denotes the Lie derivative with respect to the vector ξ . Note that this rule does not apply to Christoffel symbols as they are not tensors. For p -forms, one should insert the

expressions above and keep those terms that are linear in ξ . Then, the change in ω is given by

$$\Delta\omega = \int_{\Sigma} d\Sigma_a \sqrt{|g|} \Delta J^a. \quad (17)$$

Now, if ΔJ^a can be written as a divergence of an antisymmetric two-form i.e. $\nabla_a \mathcal{F}^{ab}$ plus terms that vanish on shell, ω is gauge invariant. The general proof for an generic gravity theory derived from an action with local symmetries is given in a corollary of [19]. However, for our purposes we will need the explicit form of \mathcal{F}^{ab} to define the conserved charges of e.g. the NMG theory.

D. Application to $\mathcal{L} \equiv \kappa^{-1}(R + 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2$

We are now ready to apply this procedure to the following quadratic action

$$I = \int d^D x \sqrt{|g|} \mathcal{L} \equiv \int d^D x \sqrt{|g|} \left(\frac{1}{\kappa} (R + 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2 \right). \quad (18)$$

The variation of (18) reads

$$\delta I = \int d^D x \sqrt{|g|} \left(\frac{1}{\kappa} \mathcal{G}_{ab} + \alpha A_{ab} + \beta B_{ab} \right) \delta g^{ab} + \int d^D x \left(\frac{1}{\kappa} \partial_a \Lambda_{\kappa}^a + \alpha \partial_a \Lambda_{\alpha}^a + \beta \partial_a \Lambda_{\beta}^a \right), \quad (19)$$

where

$$\mathcal{G}_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R - \Lambda_0 g_{ab}, \quad (20)$$

$$A_{ab} \equiv 2R R_{ab} - 2\nabla_a \nabla_b R + g_{ab} (2\Box R - \frac{1}{2} R^2), \quad (21)$$

$$B_{ab} \equiv 2R_{acbd} R^{cd} - \nabla_a \nabla_b R + \Box R_{ab} + \frac{1}{2} g_{ab} (\Box R - R_{cd} R^{cd}). \quad (22)$$

As discussed before, the boundary terms

$$\Lambda_{\kappa}^a \equiv \sqrt{|g|} \left(g^{bc} \delta \Gamma_{bc}^a - g^{ab} \delta \Gamma_{bc}^c \right), \quad (23)$$

$$\Lambda_{\alpha}^a \equiv \sqrt{|g|} \left(2R g^{bc} \delta \Gamma_{bc}^a - 2R g^{ab} \delta \Gamma_{bc}^c + 2\nabla^a R \delta \ln|g| + 2\nabla_b R \delta g^{ab} \right), \quad (24)$$

$$\Lambda_{\beta}^a \equiv \sqrt{|g|} \left(2R^{cb} \delta \Gamma_{bc}^a - 2R^{ab} \delta \Gamma_{bc}^c + \frac{1}{2} \nabla^a R \delta \ln|g| + 2\nabla_c R^a{}_b \delta g^{bc} - \nabla^a R_{cb} \delta g^{cb} \right), \quad (25)$$

yield a symplectic current given by

$$J^a = J_\kappa^a + J_\alpha^a + J_\beta^a, \quad \text{with} \quad (26)$$

$$J_\kappa^a = -\frac{\delta\Lambda_\kappa^a}{\sqrt{|g|}} = \delta\Gamma_{bc}^a \wedge (\delta g^{bc} + \frac{1}{2}g^{bc}\delta\ln|g|) - \delta\Gamma_{bc}^c \wedge (\delta g^{ab} + \frac{1}{2}g^{ab}\delta\ln|g|), \quad (27)$$

$$\begin{aligned} J_\alpha^a = -\frac{\delta\Lambda_\alpha^a}{\sqrt{|g|}} &= \delta\Gamma_{bc}^a \wedge (2R\delta g^{bc} + Rg^{bc}\delta\ln|g| + 2g^{bc}\delta R) \\ &\quad - \delta\Gamma_{bc}^c \wedge (2R\delta g^{ab} + Rg^{ab}\delta\ln|g| + 2g^{ab}\delta R) \\ &\quad - \delta\ln|g| \wedge \left(\nabla_b R \delta g^{ab} - 2\delta(\nabla^a R) \right) + \delta g^{ab} \wedge \left(2\delta(\nabla_b R) - 2\nabla_b R \delta\ln|g| \right), \end{aligned} \quad (28)$$

$$\begin{aligned} J_\beta^a = -\frac{\delta\Lambda_\beta^a}{\sqrt{|g|}} &= \delta\Gamma_{bc}^a \wedge (R^{bc}\delta\ln|g| + 2\delta R^{bc}) - \delta\Gamma_{bc}^c \wedge (R^{ab}\delta\ln|g| + 2\delta R^{ab}) \\ &\quad + \delta\ln|g| \wedge \left(\frac{1}{2}\delta(\nabla^a R) - \nabla_c R^a{}_b \delta g^{bc} + \frac{1}{2}\nabla^a R_{cb} \delta g^{bc} \right) \\ &\quad + \delta g^{bc} \wedge \left(2\delta(\nabla_c R^a{}_b) - \delta(\nabla^a R_{cb}) \right). \end{aligned} \quad (29)$$

Here, the variation of several terms such as $\delta(\nabla_c R^a{}_b)$ are quite complicated and we save the details to the Appendix. The covariant divergence of (26) vanishes on-shell as we discussed in the previous section.

There remains to investigate the gauge invariance of ω . After a cumbersome calculation, the change in the symplectic current can be written as (the transformation properties of the relevant terms are also given in the Appendix)

$$\Delta J^a = \nabla_c \mathcal{F}^{ac} + 2\Phi_{bc} \xi^c \wedge \delta g^{ab} + \Phi^a{}_c \xi^c \wedge \delta\ln|g| + \Phi_{bc} \xi^a \wedge \delta g^{bc}, \quad (30)$$

where

$$\mathcal{F}^{ac} = -\mathcal{F}^{ca} = \frac{1}{\kappa} \mathcal{F}_\kappa^{ac} + \alpha \mathcal{F}_\alpha^{ac} + \beta \mathcal{F}_\beta^{ac}, \quad (31)$$

with

$$\begin{aligned} \mathcal{F}_\kappa^{ac} &\equiv 2\xi^{[c} \wedge \nabla_b \delta g^{a]b} - 2\xi_b \wedge \nabla^{[c} \delta g^{a]b} - 2\delta g^{b[c} \wedge \nabla_b \xi^{a]} \\ &\quad - 2\xi^{[a} \wedge \nabla^{c]} \delta\ln|g| - \delta\ln|g| \wedge \nabla^{[c} \xi^{a]}, \end{aligned} \quad (32)$$

$$\mathcal{F}_\alpha^{ac} \equiv 2R\mathcal{F}_\kappa^{ac} + 4\delta g^{b[c} \wedge \xi^{a]} \nabla_b R + 4\delta R \wedge \nabla^{[a} \xi^{c]} + 8\nabla^{[c} \delta R \wedge \xi^{a]}, \quad (33)$$

$$\begin{aligned} \mathcal{F}_\beta^{ac} &\equiv 2R^{b[a} \delta\ln|g| \wedge \nabla_b \xi^{c]} + 4g^{d[a} \delta R_{de} \wedge \nabla^{b]} \xi^{c]} + 2\delta\ln|g| \wedge \xi^b \nabla^{[c} R^{a]}{}_b + 4\delta g^{d[a} \wedge \nabla_b \xi^{c]} R_d{}^b \\ &\quad - 4R_e{}^{[a} \nabla_b \xi^{c]} \wedge \delta g^{be} + 4R^{bd} \xi^{[a} \wedge \delta\Gamma^{c]}{}_{bd} + 4R^{b[a} \xi^{c]} \wedge \delta\Gamma^d{}_{bd} - 4\xi^b \wedge \delta g^{d[a} \nabla^{c]} R_{db} \\ &\quad - 4\xi_e \wedge \delta g^{b[c} \nabla_b R^{a]e} + 4g^{d[a} g^{c]e} \delta(\nabla_e R_{db}) \wedge \xi^b - 2\xi^{[a} \wedge \nabla^{c]} \delta R + 2\delta g^{b[c} \wedge \xi^{a]} \nabla_b R \\ &\quad - 2g^{bd} \xi^{[c} \wedge \nabla^{a]} \delta R_{bd} + 4g^{e[a} \xi^{c]} \wedge \nabla^b \delta R_{be} + 4g^{be} R_d{}^{[c} \xi^{a]} \wedge \delta\Gamma^d{}_{be} + 4R^{d[a} \delta\Gamma^{c]}{}_{bd} \wedge \xi^b. \end{aligned} \quad (34)$$

The first term in (30) vanishes when inserted in the integral in (8) for sufficiently fast decaying metric variations and the remaining terms vanish on-shell. In the next section, we will discuss how conserved charges can be obtained from (31) and will derive an equality relating the ADT charge definition [5–7] and the charge expression obtained from the symplectic two-form.

III. THE CONSERVED CHARGES

In a recent work [9], the conserved charges of the TMG were obtained from the change in the symplectic current given in (17) under the group of diffeomorphisms. Here we use the same idea to show that the charge expressions obtained in this formalism and the ADT charge [5–7, 20] are equivalent for theories derived from a local gravity action. We first consider the transformation of (6) under (13)

$$\begin{aligned} & -2 \int d^D x \sqrt{|g|} \delta \Phi_{ab} \wedge \nabla^a \xi^b + \int d^D x \sqrt{|g|} \mathcal{L}_\xi \Phi_{ab} \wedge \delta g^{ab} - \frac{1}{2} \int d^D x \sqrt{|g|} \Phi_{ab} \delta g^{ab} \wedge \nabla_c \xi^c \\ & + \int d^D x \sqrt{|g|} \Phi_{ab} \nabla^a \xi^b \wedge \delta \ln |g| - \int d^D x \sqrt{|g|} \nabla_a (\Delta J^a) = 0, \end{aligned} \quad (35)$$

where ΔJ^a is the change in symplectic current. The first term in (35) can be cast as a divergence since $\nabla_a \delta \Phi^{ab} = 0$, which follows from the Bianchi identity. Thus, we obtain

$$\begin{aligned} \int d^D x \sqrt{|g|} \nabla_a (2 \delta \Phi^{ab} \wedge \xi_b + \Delta J^a) &= \int d^D x \sqrt{|g|} \Phi_{ab} \nabla^a \xi^b \wedge \delta \ln |g| \\ &\quad - \int d^D x \sqrt{|g|} \delta g^{ab} \wedge (\mathcal{L}_\xi \Phi_{ab} + \frac{1}{2} \Phi_{ab} \nabla_c \xi^c). \end{aligned} \quad (36)$$

We now further restrict our attention to the case where the metric is linearized as $g_{ab} = \bar{g}_{ab} + h_{ab}$, and the deviation h_{ab} should vanish sufficiently slow as one approaches the background \bar{g}_{ab} at “infinity”². We also assume that the background spacetime \bar{g}_{ab} admits a globally defined Killing vector $\bar{\xi}_a$. Indices are raised/lowered and covariant derivatives are defined with respect to the background metric \bar{g}_{ab} as usual. The variation is identified as $\delta g_{ab} \rightarrow h_{ab}$, $\delta g^{ab} \rightarrow -h^{ab}$. Therefore, the terms like R_{ab} , R are identified with the background ones \bar{R}_{ab} , \bar{R} and the terms like $\delta(\nabla_a R_{bc})$ are taken as $(\nabla_a R_{bc})_L$, where subscript L indicates the linearized version of the corresponding quantity. Finally, we put all of the ξ terms into the

² Here this condition guarantees nonzero results for the conserved charge.

right hand side of the wedge products and then drop them. With all of these identifications and by the help of field equations, (36) yields

$$\int d^D x \sqrt{|\bar{g}|} \bar{\nabla}_a ((\Phi^{ab})_L \bar{\xi}_b) = -\frac{1}{2} \int d^D x \sqrt{|\bar{g}|} \bar{\nabla}_a (\Delta \tilde{J}^a). \quad (37)$$

where $\Delta \tilde{J}^a$ is the vector obtained from the two-form ΔJ^a after identifications. The left hand side of (37) is the conserved current which is used to construct the ADT [5] charge.³ From this we obtain the charge expression as⁴

$$Q_{ADT}(\bar{\xi}) = -\frac{1}{2} \int_{\Sigma} d^{D-1} x \sqrt{|\sigma|} n_a \bar{\nabla}_c Q^{ac} = -\frac{1}{2} \int_{\partial \Sigma} d^{D-2} x \sqrt{|\sigma^{(\partial \Sigma)}|} n_a s_c Q^{ac}, \quad (38)$$

where

$$Q^{ac} = -Q^{ca} = \frac{1}{\kappa} Q_{\kappa}^{ac} + \alpha Q_{\alpha}^{ac} + \beta Q_{\beta}^{ac}, \quad (39)$$

$$Q_{\kappa}^{ac} \equiv 2 \bar{\nabla}_b h^{b[a} \bar{\xi}^{c]} - 2 \bar{\nabla}^{[c} h^{a]b} \bar{\xi}_b + 2 h^{b[c} \bar{\nabla}_b \bar{\xi}^{a]} + 2 (\bar{\nabla}^{[c} h) \bar{\xi}^{a]} - h \bar{\nabla}^{[c} \bar{\xi}^{a]}, \quad (40)$$

$$Q_{\alpha}^{ac} \equiv 2 \bar{R} Q_{\kappa}^{ac} - 4 \bar{\nabla}_b \bar{R} h^{b[c} \bar{\xi}^{a]} + 4 R_L \bar{\nabla}^{[a} \bar{\xi}^{c]} + 8 (\bar{\nabla}^{[c} R_L) \bar{\xi}^{a]}, \quad (41)$$

$$\begin{aligned} Q_{\beta}^{ac} \equiv & 2 \bar{R}^{b[a} h \bar{\nabla}_b \bar{\xi}^{c]} + 4 \bar{g}^{d[a} (R_{de})_L \bar{\nabla}^{b]c} \bar{\xi}^d + 2 \bar{\nabla}^{[c} \bar{R}^{a]}_b h \bar{\xi}^b - 4 h^{d[a} \bar{\nabla}_b \bar{\xi}^{c]} \bar{R}_d{}^b \\ & + 4 \bar{R}^{e[a} h_{be} \bar{\nabla}^{c]} \bar{\xi}^b - 4 \bar{R}^{bd} (\Gamma^{[c}_{bd})_L \bar{\xi}^{a]} - 4 \bar{R}^{b[a} (\Gamma^{d]}_{bd})_L \bar{\xi}^{c]} - 4 h^{d[a} \bar{\xi}^{b]c} \bar{\nabla}^{d]} \bar{R}_{db} \\ & - 4 h^{b[c} \bar{\xi}_e \bar{\nabla}_b \bar{R}^{a]e} + 4 \bar{g}^{d[a} \bar{g}^{c]e} (\nabla_e R_{db})_L \bar{\xi}^b + 2 (\bar{\nabla}^{[c} R_L) \bar{\xi}^{a]} - 2 h^{b[c} \bar{\xi}^{a]} \bar{\nabla}_b \bar{R} \\ & + 2 \bar{g}^{bd} \bar{\nabla}^{[a} (R_{bd})_L \bar{\xi}^{c]} - 4 \bar{g}^{e[a} \bar{\nabla}^{b]c} (R_{be})_L \bar{\xi}^c - 4 \bar{g}^{be} \bar{R}_d{}^{[c} (\Gamma^{d]}_{be})_L \bar{\xi}^{a]} + 4 \bar{R}^{d[a} (\Gamma^{c]}_{bd})_L \bar{\xi}^b. \end{aligned} \quad (42)$$

The first two of the charge expressions (40) and (41) are identical to their counterparts given in [20], the equivalence of the third one can be shown after some computation. The next section is devoted to the calculation of the conserved charges of some solutions of NMG using this expression.

IV. THE CONSERVED CHARGES OF SOME SOLUTIONS OF NMG

Having found the charge expression (38), let us consider some black hole solutions of NMG for which we can use (38) to compute the conserved charges. First we work out the examples that are asymptotically AdS₃, e.g. the BTZ blackhole [14] and the solutions given

³ This relation accounts for the factor of $-1/2$ used in [9].

⁴ Σ is a $(D-1)$ -dimensional spacelike hypersurface with induced metric σ and unit normal vector n^a , $\partial \Sigma$ (boundary of Σ) is a $(D-2)$ -dimensional hypersurface with induced metric $\sigma^{(\partial \Sigma)}$ and unit normal s^c .

in [17, 18]. Then we consider the solutions with asymptotes that are not spaces of constant curvature, namely the three-dimensional Lifshitz black hole [15] and the warped AdS_3 black hole given in [16]. Both examples have been studied in [16, 21, 22] with which we compare the results.

A. The BTZ black hole

The first example is the celebrated BTZ black hole [14], which can be cast in the form

$$ds^2 = \left(\frac{-2\rho}{l^2} + \frac{M}{2} \right) dt^2 + \left(\frac{4\rho^2}{l^2} - \frac{(M^2 l^2 - J^2)}{4} \right)^{-1} d\rho^2 - J dt d\phi + \left(2\rho + \frac{M l^2}{2} \right) d\phi^2, \quad (43)$$

and this is a solution of NMG when

$$\kappa = 16\pi G, \quad \beta = -\frac{1}{\kappa m^2}, \quad \Lambda_0 = \frac{1 + 4l^2 m^2}{4l^4 m^2}, \quad \alpha = -\frac{3}{8}\beta. \quad (44)$$

Here m^2 is a “relative” mass parameter of the NMG [10]. The background spacetime is taken to be AdS_3 that is obtained by setting $M \rightarrow 0, J \rightarrow 0$ in (43)

$$ds^2 = -\frac{2\rho}{l^2} dt^2 + \frac{l^2}{4\rho^2} d\rho^2 + 2\rho d\phi^2. \quad (45)$$

This form of AdS_3 clearly possesses two globally defined Killing vectors $\bar{\xi}^a = (-1, 0, 0)$ and $\bar{\vartheta}^a = (0, 0, 1)$ that are used in the computation of the energy and angular momentum, respectively. The timelike and spacelike normals that follow from the normalization condition i.e. $n^a n_a = -1, s^a s_a = +1$ are

$$n^a = -\frac{\ell}{\sqrt{2\rho}} \delta_t^a, \quad s^a = \frac{2\rho}{\ell} \delta_\rho^a.$$

Finally the measure of (38) is simply $\sqrt{|\sigma^{(\partial\Sigma)}|} = \sqrt{2\rho}$. The conserved charges are obtained using (38)

$$E_{BTZ} = \left(1 - \frac{1}{2l^2 m^2} \right) \frac{M}{16G}, \quad J_{BTZ} = \left(1 - \frac{1}{2l^2 m^2} \right) \frac{J}{16G}. \quad (46)$$

These “renormalized mass and angular momentum” coincide with the ones given in [16] that employed ADT charge definition for computation and [22] in which the boundary stress tensor method was used.

B. The “logarithmic” black hole in [18]

As a second example consider the black hole solution given in [18]

$$ds^2 = -\frac{4\rho^2}{\ell^2 f(\rho)} dt^2 + f(\rho) \left[d\phi - \frac{q\ell \ln |\rho/\rho_0|}{f(\rho)} dt \right]^2 + \frac{\ell^2 d\rho^2}{4\rho^2}, \quad (47)$$

where

$$f(\rho) = 2\rho + q\ell^2 \ln |\rho/\rho_0|.$$

This is a solution to the NMG with

$$\kappa = 8\pi G, \quad \beta = -\frac{2\ell^2}{\kappa}, \quad \Lambda_0 = \frac{3}{2\ell^2}. \quad (48)$$

The background spacetime is taken to be AdS_3 in the form (45), therefore we can employ the same Killing vectors, normals and induced metric as in the BTZ case. Following the same lines for the calculation of charges, we find

$$E = \lim_{\rho \rightarrow \infty} \int_0^{2\pi} \sqrt{2\rho} n_t s_\rho Q^{t\rho}(\bar{\xi}) d\phi = \frac{2q}{G}, \quad (49)$$

$$J = \lim_{\rho \rightarrow \infty} \int_0^{2\pi} \sqrt{2\rho} n_t s_\rho Q^{t\rho}(\bar{\vartheta}) d\phi = \frac{2\ell q}{G}. \quad (50)$$

This result is identical to the one given in [18] that was again computed through ADT.

C. The rotating black hole in [17]

The next example that is of interest is the stationary solution given in [17]

$$ds^2 = \left(-N(r)F(r) + r^2 K(r)^2 \right) dt^2 + \frac{dr^2}{F(r)} + 2r^2 K(r) dt d\phi + r^2 d\phi^2, \quad (51)$$

where

$$N(r) = \left[1 + \frac{q\ell^2}{4H(r)} (1 - \sqrt{\Xi}) \right]^2, \quad (52)$$

$$F(r) = \frac{H(r)^2}{r^2} \left[\frac{H(r)^2}{\ell^2} + \frac{q}{2} (1 + \sqrt{\Xi}) H(r) + \frac{q^2 \ell^2}{16} (1 - \sqrt{\Xi})^2 - 4GM\sqrt{\Xi} \right], \quad (53)$$

$$K(r) = -\frac{p}{2r^2} (4GM - qH(r)), \quad (54)$$

$$H(r) = \left[r^2 - 2GM\ell^2 (1 - \sqrt{\Xi}) - \frac{q^2 \ell^4}{16} (1 - \sqrt{\Xi})^2 \right]^{1/2}, \quad (55)$$

$$\Xi \equiv 1 - p^2/\ell^2, \quad (56)$$

with

$$\Lambda_0 = \frac{1}{2\ell^2}, \quad \beta = \frac{2\ell^2}{\kappa}, \quad \alpha = -\frac{3}{8}\beta, \quad \kappa = 16\pi G.$$

in our conventions. The rotation parameter p is restricted between $-\ell \leq p \leq \ell$ and the parameter q is the additional “gravitational hair” for which the $b = 0$ case is the BTZ blackhole. The background spacetime relevant for our purposes can be found by setting $q \rightarrow 0$, $M \rightarrow 0$ in (51) that is simply AdS₃ spacetime

$$ds^2 = -\frac{r^2}{\ell^2}dt^2 + \frac{\ell^2}{r^2}dr^2 + r^2d\phi^2. \quad (57)$$

The timelike and spacelike normals are

$$n^a = -\frac{\ell}{r}\delta_t^a, \quad s^a = \frac{r}{\ell}\delta_r^a, \quad \sigma^{(\partial\Sigma)} = r^2.$$

With those choices we compute the energy and angular momentum to be

$$E = \lim_{r \rightarrow \infty} \int_0^{2\pi} r n_t s_r Q^{tr}(\bar{\xi}) d\phi = M, \quad (58)$$

$$J = \lim_{r \rightarrow \infty} \int_0^{2\pi} r n_t s_r Q^{tr}(\bar{\vartheta}) d\phi = Mp. \quad (59)$$

As discussed in [17], the parameter b does not appear in the conserved charges, which is the reason it was called “gravitational hair” in the first place.

D. Three-dimensional Lifshitz black hole

The first example with a nonconstant curvature background is the three-dimensional Lifshitz black hole [15] given as

$$ds^2 = -\frac{r^6}{\ell^6} \left(1 - \frac{M\ell^2}{r^2}\right) dt^2 + \frac{\ell^2}{r^2} \left(1 - \frac{M\ell^2}{r^2}\right)^{-1} dr^2 + \frac{r^2}{\ell^2} dx^2, \quad (60)$$

which solves NMG with

$$\Lambda_0 = \frac{13}{2\ell^2}, \quad \beta = \frac{2\ell^2}{\kappa}, \quad \alpha = -\frac{3\ell^2}{4\kappa}, \quad \kappa = 16\pi G.$$

The background metric can be obtained by taking $M \rightarrow 0$

$$ds^2 = -\frac{r^6}{\ell^6} dt^2 + \frac{\ell^2}{r^2} dr^2 + \frac{r^2}{\ell^2} dx^2.$$

The timelike, spacelike normals and one-dimensional induced metric can easily be found as

$$n^a = -\frac{\ell^3}{r^3}\delta_t^a, \quad s^a = \frac{r}{\ell}\delta_r^a, \quad \sigma^{(\partial\Sigma)} = \frac{r^2}{\ell^2}.$$

For the energy, the timelike Killing vector $\bar{\xi}^a = -\delta_t^a$ can be employed. With these, (38) yields

$$E = \lim_{r \rightarrow \infty} \int_0^{2\pi\ell} \frac{r}{\ell} n_t s_r Q^{tr}(\bar{\xi}) dx = \frac{7M^2}{8G}. \quad (61)$$

That is the same energy given in [20] that was calculated through the ADT procedure for arbitrary backgrounds, yet the result differs from the expression in [22].

E. The Warped AdS₃ black hole

The final example is the warped AdS₃ black hole [16] that reads

$$ds^2 = -\mu^2 \frac{r^2 - r_0^2}{F(r)} dt^2 + F(r) \left[d\phi - \frac{r + (1 - \mu^2)\omega}{F(r)} dt \right]^2 + \frac{1}{\mu^2 \zeta^2} \frac{dr^2}{r^2 - r_0^2}, \quad (62)$$

where

$$F(r) = r^2 + 2\omega r + \omega^2(1 - \mu^2) + \frac{\mu^2 r_0^2}{1 - \mu^2}.$$

This is a solution of the NMG theory with

$$\kappa = 8\pi G, \quad \beta = -\frac{1}{m^2 \kappa}, \quad \alpha = \frac{3}{8m^2 \kappa},$$

$$\mu^2 = \frac{9m^2 + 21\Lambda_0 - 2m\sqrt{3(5m^2 - 7\Lambda_0)}}{4(m^2 + \Lambda_0)} \quad \text{and} \quad \zeta^2 = \frac{8m^2}{21 - 4\mu^2}$$

with m^2 as the NMG parameter. In order to have a causally regular black hole, μ^2 and Λ_0 must be [16]

$$0 < \mu^2 < 1 \quad \text{and} \quad \frac{35m^2}{289} \geq \Lambda_0 \geq -\frac{m^2}{21}.$$

The background spacetime of this black hole can be defined by taking $\omega \rightarrow 0$, $r_0 \rightarrow 0$ in (62)

$$ds^2 = (1 - \mu^2) dt^2 + \frac{1}{r^2 \zeta^2 \mu^2} dr^2 - 2r d\phi dt + r^2 d\phi^2. \quad (63)$$

The timelike, spacelike normals and the measure is apparent considering the standard ADM form of the metric (63)

$$n_a = -\mu \delta_a^t, \quad s_a = \frac{1}{\mu r \zeta} \delta_a^r, \quad \sqrt{|\sigma^{(\partial\Sigma)}|} = r.$$

To find the energy, one again has to choose the timelike Killing vector as $\bar{\xi}^a = -\delta_t^a$ and for the angular momentum one has to use $\bar{\vartheta}^a = \delta_\phi^a$. Then (38) yields

$$E = \lim_{r \rightarrow \infty} \int_0^{2\pi} r n_t s_r Q^{tr}(\bar{\xi}) d\phi = \frac{4\mu^2(1 - \mu^2)\omega\zeta}{G(21 - 4\mu^2)}, \quad (64)$$

$$J = \lim_{r \rightarrow \infty} \int_0^{2\pi} r n_t s_r Q^{tr}(\bar{\vartheta}) d\phi = -\frac{\zeta}{8G(21 - 4\mu^2)} \left[\frac{16r_0^2\mu^2}{(1 - \mu^2)} + \frac{(1 - \mu^2)}{\mu^2} (21 - 29\mu^2 + 24\mu^4)\omega^2 \right].$$

The values for the energy and angular momentum agrees with the ones given in [21], however angular momentum is in conflict with the one in [16]. The discrepancy of these results, and the validity of the charge expression are discussed more explicitly in [21].

V. CONCLUSIONS

In this paper, starting from a local gravity action described by (2), we showed that a covariantly conserved symplectic current can always be obtained from the boundary terms that appear in the first variation of the action. Moreover, we have shown that the two-form obtained from the integration of the symplectic current over a spacelike hypersurface is closed for any theory. The investigation of the gauge invariance of this two-form is the final task that needs to be performed in order to show that it is the symplectic two-form of the theory, which provides the most important result of this paper. Under diffeomorphisms, the symplectic two-form of a generic gravity theory yields a conserved Killing charge expression that is equivalent to the extended ADT formalism for arbitrary backgrounds with at least one global Killing isometry [20].

As a consistency check, we obtained the charge expression for NMG and calculated the energy and angular momentum of several black holes. The charges of black holes with AdS_3 backgrounds are in agreement with the previous works [16, 17, 22]. In the case of nonconstant curvature backgrounds, namely, Lifshitz and warped AdS_3 spacetimes, the results agree with the ones computed through the ADT procedure for arbitrary backgrounds [20, 21], which was expected since the charge expressions were shown to be covariantly equivalent. On the other hand, it was shown in [21] that in the case of Lifshitz and warped AdS_3 black holes, the charge expressions obtained were in conflict with the results found by other means [16, 22]. This discrepancy calls for further study regarding the validity of this charge expression for generic backgrounds. It would also be interesting to perform a covariant, geometric quantization of the theories described by (18).

Appendix:

Here we first present the identities that are used to compute the variation of a covariant derivative of a tensor. Then, we list the transformation of terms that we have used during

the calculation of $\Delta\omega$.

From the well known equality of Christoffel symbols

$$\delta\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}(\nabla_b\delta g_{cd} + \nabla_c\delta g_{bd} - \nabla_d\delta g_{bc}). \quad (\text{A.1})$$

variation of the Riemann tensor can be calculated simply

$$\delta R^a{}_{bcd} = \nabla_c\delta\Gamma^a{}_{bd} - \nabla_d\delta\Gamma^a{}_{bc}, \quad (\text{A.2})$$

and the contraction of indices leads to

$$\delta R_{ab} = \nabla_c\delta\Gamma^c{}_{ab} - \nabla_b\delta\Gamma^c{}_{ac}. \quad (\text{A.3})$$

After a straightforward calculation the variation of the covariant derivative of a tensor $T_{b\dots}{}^{c\dots}$ can be written as

$$\delta(\nabla_a T_{b\dots}{}^{c\dots}) = \nabla_a\delta T_{b\dots}{}^{c\dots} - \delta\Gamma^i{}_{ab} T_{i\dots}{}^{c\dots} - \dots + \delta\Gamma^c{}_{ai} T_{b\dots}{}^{i\dots} + \dots, \quad (\text{A.4})$$

which is reminiscent of the usual covariant derivative formula where Christoffel symbols are replaced with $\delta\Gamma^a{}_{bc}$. Application of this formula together with the identities above leads to the following useful relations

$$\begin{aligned} \delta(\nabla_a R_{cd}) &= \nabla_a\delta R_{cd} - R_{ed}\delta\Gamma^e{}_{ac} - R_{ec}\delta\Gamma^e{}_{ad}, \\ \delta(\nabla_b\nabla_a R_{cd}) &= \nabla_b\delta(\nabla_a R_{cd}) - \nabla_e R_{cd}\delta\Gamma^e{}_{ba} - \nabla_a R_{ed}\delta\Gamma^e{}_{bc} - \nabla_a R_{ce}\delta\Gamma^e{}_{bd}, \\ \delta(\square R_{cd}) &= g^{ab}\delta(\nabla_b\nabla_a R_{cd}) + \nabla_b\nabla_a R_{cd}\delta g^{ab}, \\ \delta(\nabla_b\nabla_a R) &= g^{cd}\delta(\nabla_b\nabla_a R_{cd}) + \nabla_b\nabla_a R_{cd}\delta g^{cd}, \\ \delta(\square R) &= g^{ab}\delta(\nabla_b\nabla_a R) + \nabla_b\nabla_a R\delta g^{ab}. \end{aligned} \quad (\text{A.5})$$

Throughout the calculation of $\Delta\omega$, the transformation of the following terms under $\delta g_{ab} \rightarrow \delta g_{ab} + \nabla_a\xi_b + \nabla_b\xi_a$ has also been used:

$$\begin{aligned} \delta\Gamma^a{}_{bc} &\rightarrow \delta\Gamma^a{}_{bc} + R_{ec}{}^a{}_b\xi^e + \nabla_c\nabla_b\xi^a, \\ \delta\ln|g| &\rightarrow \delta\ln|g| + 2\nabla_a\xi^a, \\ \delta R &\rightarrow \delta R + \xi^a\nabla_a R, \\ \delta(\nabla_a R) &\rightarrow \delta(\nabla_a R) + \nabla_a\nabla_b R\xi^b + \nabla_b R\nabla^b\xi_a, \\ \delta R_{ab} &\rightarrow \delta R_{ab} + \nabla_c R_{ab}\xi^c + R_{ad}\nabla_b\xi^d + R_{bd}\nabla_a\xi^d. \end{aligned} \quad (\text{A.6})$$

As stated previously, the change in the variation of a tensor is given by the Lie derivative of that tensor along ξ .

Acknowledgments

We are grateful to Özgür Sarioğlu for his guidance and critical reading of the manuscript. We also thank Bayram Tekin for his useful comments. This work is partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK). GA is supported by TÜBİTAK MSc Scholarship.

- [1] R. L. Arnowitt, S. Deser and C. W. Misner, gr-qc/0405109.
- [2] M.J. Gotay, in *Differential Geometric Methods in Theoretical Physics*, K. Bleuler & M. Werner, Eds., NATO Advanced Study Institutes, Series C., Vol. 250 (Kluwer Acad. Publ., Dordrecht 1988), 295.
- [3] E. Witten Nuc. Phys. B **276** 291 (1986); C. Crnkovic and E. Witten in *Three hundred years of gravitation* S. W. Hawking and W. Israel, eds. Cambridge University Press (1987).
- [4] G.J. Zuckerman in *Mathematical Aspects of String Theory* S.T. Yau ed. World Scientific Singapore (1987).
- [5] S. Deser and B. Tekin, Phys. Rev. Lett. **89**, 101101 (2002) [arXiv:hep-th/0205318].
- [6] S. Deser and B. Tekin, Phys. Rev. D **67**, 084009 (2003) [arXiv:hep-th/0212292].
- [7] L. F. Abbott and S. Deser, Nucl. Phys. B **195**, 76 (1982).
- [8] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Annals Phys. **140**, 372 (1982).
- [9] C. Nazaroglu, Y. Nutku, B. Tekin, Phys. Rev. **D83**, 124039 (2011). [arXiv:1104.3404 [hep-th]].
- [10] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009) [arXiv:0901.1766 [hep-th]].
- [11] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. D **79**, 124042 (2009) [arXiv:0905.1259 [hep-th]].
- [12] I. Gullu and B. Tekin, Phys. Rev. D **80**, 064033 (2009) [arXiv:0906.0102 [hep-th]].
- [13] K. Muneyuki and N. Ohta, arXiv:1201.2058 [hep-th].
- [14] M. Banados, C. Teitelboim, J. Zanelli, Phys. Rev. Lett. **69**, 1849-1851 (1992). [hep-th/9204099].
- [15] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, Phys. Rev. D **80**, 104029 (2009)

- [arXiv:0909.1347 [hep-th]].
- [16] G. Clement, Class. Quant. Grav. **26**, 105015 (2009). [arXiv:0902.4634 [hep-th]].
 - [17] J. Oliva, D. Tempo, R. Troncoso, JHEP **0907**, 011 (2009). [arXiv:0905.1545 [hep-th]].
 - [18] G. Clement, Class. Quant. Grav. **26**, 165002 (2009). [arXiv:0905.0553 [hep-th]].
 - [19] J. Lee and R. M. Wald, J. Math. Phys. **31**, 725 (1990).
 - [20] D. O. Devecioglu and O. Sarioglu, Phys. Rev. D **83**, 021503 (2011) [arXiv:1010.1711 [hep-th]].
 - [21] D. O. Devecioglu, O. Sarioglu, Phys. Rev. **D83**, 124041 (2011). [arXiv:1103.1993 [hep-th]].
 - [22] O. Hohm and E. Tonni, JHEP **1004**, 093 (2010) [arXiv:1001.3598 [hep-th]].